Strategy for Integration

Firstly, try to simplify the integrand if possible. For example:

$$\int \sqrt{x}(1+\sqrt{x})\,dx = \int (\sqrt{x}+x)\,dx, \quad \int \frac{\tan\theta}{\sec^2\theta}\,d\theta = \int \frac{\sin\theta}{\cos\theta}\cos^2\theta\,d\theta, \cdots$$

Usually, the following **5** methods can cover all the integration problems.

1. ***Integration Formulas*:** Some elementary functions, for example:

$$\int x^n\,dx\,(n\neq -1), \int \frac{1}{x}\,dx, \quad \int \sin x\,dx, \int \sec x\,dx, \quad \int e^x\,dx, \quad \int \frac{1}{1+x^2}\,dx, \cdots$$

2. u-***Substitution*:** Some function $\underline{u=g(x)}$ and $\underline{du=g'(x)dx}$ show up at the same time. For example:

$$\int \frac{x}{x^2-1}\,dx, \int \frac{x}{\sqrt{x^2-1}}\,dx, \quad \int \sin^m x\cos^n x\,dx, \quad \int xe^{x^2}\,dx, \quad \int \frac{\ln x}{x}\,dx, \cdots$$

3. ***Integration by parts*:** Usually two different types of functions show up at the same time. And one of them usually is the power of x. e.g:

$$\int x\sin x\,dx, \int x\sin^m x\cos^n x\,dx, \quad \int x^2e^x\,dx, \quad \int x\ln x\,dx, \int \ln x\,dx, \cdots$$

4. ***Rational functions*** $\frac{P(x)}{Q(x)}$**:** The key method is partial fractions. For this case, just be careful of the algebraic calculation.

5. ***Radicals*:** Usually there two types of questions in this case:

   (a) **Trigonometric substitution:** To deal with something like $\sqrt{\pm x^2 \pm a^2}$.

$$\int \frac{\sqrt{a^2-x^2}}{x^2}\,dx, \quad \int \frac{1}{\sqrt{x^2-a^2}}\,dx, \quad \int \frac{x^3}{\sqrt{x^2+a^2}}\,dx, \cdots$$

   (b) **Rationalizing substitution:** To deal with something like $\sqrt[n]{ax+b}$ or sometimes even for more general $\sqrt[n]{g(x)}$. For example,

$$\int \sqrt{\frac{1-x}{1+x}}\,dx, \quad \int x\sqrt[3]{4x+3}\,dx, \int x^2\sqrt{2+x}\,dx \cdots$$

Some Identities and Some common mistakes happened in the quiz

1. Summation formulas: For $\sum_{k=1}^{n} k$, $\sum_{k=1}^{n} k^2$ and $\sum_{k=1}^{n} k^3$, you can find them.

$$\sum_{k=1}^{n} 1 = \underbrace{1 + 1 + \cdots + 1}_{n} = n, \quad \sum_{k=1}^{n} \frac{1}{n} = \underbrace{\frac{1}{n} + \frac{1}{n} + \cdots + \frac{1}{n}}_{n} = n \cdot \frac{1}{n} = 1$$

2. Keep <u>One property of the integral</u> in mind: $\int f \pm g\, dx = \int f\, dx \pm \int g\, dx$

   For example: $\int \frac{t^3-1}{t^2}\, dt = \int \left(\frac{t^3}{t^2} - \frac{1}{t^2}\right) dt = \int t\, dt - \int t^{-2}\, dt = \frac{t^2}{2} + \frac{1}{t} + C$

3. <u>Don't forget</u> the **constant term** C in the Indefinite Integrals.

4. <u>Completing the square.</u>$(a \pm b)^2 = a^2 \pm 2 \cdot a \cdot b + b^2$

   For example, $x^2 + 2x - 8 = (x^2 + 2 \cdot x \cdot 1 + 1^2) - 1^2 - 8 = (x+1)^2 - 9$.

5. **Trigonometric Identities:** (See the **shadow part** in the below graph.)

$$\begin{aligned} \sin^2 x + \cos^2 x &= 1 \\ \tan^2 x + 1 &= \sec^2 x \\ 1 + \cot^2 x &= \csc^2 x \end{aligned}$$

6. **Double Angle Formula:**

$$\cos(2x) = \cos^2 x - \sin^2 x \Longrightarrow \begin{cases} \cos^2 x = \frac{1+\cos(2x)}{2} \\ \sin^2 x = \frac{1-\cos(2x)}{2} \end{cases}$$

$$\sin(2x) = 2 \sin x \cos x$$

7. **Volume:** $\int A\, dx$ (or $\int A\, dy$)

   (a) <u>**Disk Method**</u>: $A = \pi R^2$ (Special case of Washer Method, $r = 0$);

   (b) <u>**Washer Method**</u>: $A = \pi R^2 - \pi r^2$;

   (c) <u>**Shell Method**</u>: $A = 2\pi r \cdot h$.

   Firstly, try to **draw** the graph of the function and be careful of the <u>rotation line</u>. Then, solve the equations to find the **intersection points** to <u>determine the domain</u> of the integration. At last, find <u>R, r, h</u>**!**